Dimension theory for families of sets

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- How to obtain dimension theory for families of sets?
- Why dimension theory: to obtain definability hierarchies according to the dimension.
- First order operations should not increase the dimension i.e. everything definable from something of dimension n should have dimension at most n.
- So the dimension should come from what you add to first order logic.
- You can add a generalized quantifier in order to make a model class definable.
- We define dimension so that even generalized (Lindström) quantifiers do not change it.
- As a results, we obtain very strong hierarchy results.

The background

- Ciardelli defined in his Master's Thesis [Cia09] a dimension concept, in the case of downward closed families.
- Hella, Luosto, Sano and Virtema [HLSV14] introduced a similar dimension concept in modal logic.
- Hella and Stumpf [HS15] used a form of dimension to prove a succinctness result for the inclusion atom in modal inclusion logic.
- Lück and Vilander [LV19] generalized the notion of dimension from downward closed families to arbitrary families in the context of propositional logic.

Other dimensions

- Matroid rank: Our families do not necessarily satisfy the Exchange Axiom of matroids and therefore this concept does not work in our context.
- Vapnik–Chervonenkis- or VC-dimension is not preserved by logical operations in the sense that our dimension is.

- A family of the form $[A, B] = \{C \mid A \subseteq C \subseteq B\}$ is called an *interval*.
- The family $\mathcal{A}$ is *convex* if for all $S, T \in \mathcal{A}$, we have $[S, T] \subseteq \mathcal{A}$.
- A family of set $\mathcal{A}$ is *dominated* *(by $\bigcup \mathcal{A}$)* if $\bigcup \mathcal{A} \in \mathcal{A}$.

Dimension

- Let $\mathcal{A}$ be a family of sets. We say that a subfamily $\mathcal{G} \subseteq \mathcal{A}$ *dominates* $\mathcal{A}$ if there exist dominated convex families $\mathcal{A}_G$, $G \in \mathcal{G}$, such that $\bigcup_{G\in\mathcal{G}} \mathcal{A}_G = \mathcal{A}$ and $\bigcup \mathcal{A}_G = G$, for each $G \in \mathcal{G}$.
- The *dimension* of the family is $\mathcal{A}$

$$\mathrm{D}(\mathcal{A}) = \min\{|\mathcal{G}| \mid \mathcal{G} \text{ dominates the family } \mathcal{A}\},$$

- We consider operators: $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$.
- The union operator $\Delta_{\cup}^{X}\colon \mathcal{P}(\mathcal{P}(X))^2 \to \mathcal{P}(\mathcal{P}(X))$ is defined by $\Delta_{\cup}^{X}(\mathcal{A},\mathcal{B}) = \mathcal{A} \cup \mathcal{B}$.
- The intersection operator $\Delta_{\cap}^{X}\colon \mathcal{P}(\mathcal{P}(X))^2 \to \mathcal{P}(\mathcal{P}(X))$ is defined by $\Delta_{\cap}^{X}(\mathcal{A},\mathcal{B}) = \mathcal{A} \cap \mathcal{B}$.
- Complementation is the unary operator $\Delta_{c}^{X}\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(X))$ defined by $\Delta_{c}^{X}(\mathcal{A}) = \mathcal{P}(X) \setminus \mathcal{A}$.
- The idea of tensor disjunction $\Delta_{\vee}^{X}$ and tensor conjunction $\Delta_{\wedge}^{X}$ is to take unions and intersections inside the families: $\Delta_{\vee}^{X}(\mathcal{A},\mathcal{B}) = \{A \cup B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$ and $\Delta_{\wedge}^{X}(\mathcal{A},\mathcal{B}) = \{A \cap B \mid A \in \mathcal{A}, B \in \mathcal{B}\}$.

- Pushing complementation inside a given family, we obtain tensor negation: $\Delta_{\neg}^{X}(\mathcal{A}) = \{X \setminus A \mid A \in \mathcal{A}\}$.
- Let $f \colon X \to Y$ be a surjective function. The (abstract) projection operator corresponding to f is obtained by lifting f to a function $\Delta_f \colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(Y))$ in the usual way: $\Delta_f(\mathcal{A}) = \{f[A] \mid A \in \mathcal{A}\}$, where $f[A]$ denotes the image $\{f(a) \mid a \in A\}$ of A under f.
- Given a surjection $f \colon X \to Y$, we can also define a useful operator $\Delta_{f^{-1}} \colon \mathcal{P}(\mathcal{P}(Y)) \to \mathcal{P}(\mathcal{P}(X))$ as follows: $\Delta_{f^{-1}}(\mathcal{B}) = \{A \in \mathcal{P}(X) \mid f[A] \in \mathcal{B}\}$.

- Consider the concrete projection function $f\colon X \to Y$ for $X = X_0 \times \cdots \times X_{m-1}$ and $Y = X_0 \times \cdots X_{i-1} \times X_{i+1} \times \cdots \times X_{m-1}$ defined by $f(a_0, \ldots, a_{m-1}) = (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m-1})$ (i.e., f is the projection to coordinates $j \neq i$).
- Thus, Δ_f corresponds to the logical operation of existential quantification, and accordingly we denote it by $\Delta^X_{\exists i}$.
- Similarly, we define an operator $\Delta^X_{\forall i}\colon \mathcal{P}(\mathcal{P}(X)) \to \mathcal{P}(\mathcal{P}(Y))$ that corresponds to universal quantification: Given a set $B \in \mathcal{P}(Y)$, let $B[X_i/i] = \{(a_0, \ldots, a_{m-1}) \in X \mid (a_0, \ldots, a_{i-1}, a_{i+1}, \ldots, a_{m-1}) \in B, a_i \in X_i\}$. Then we let $\Delta^X_{\forall i}(\mathcal{A}) = \{B \in \mathcal{P}(Y) \mid B[X_i/i] \in \mathcal{A}\}$.

Note that the union and intersection operators $\Delta_{\cup}^{X}$ and $\Delta_{\cap}^{X}$ do not depend on the base set X. Thus, in the sequel we will denote these operators simply by $\cup$ and $\cap$. The same holds for tensor disjunction and conjunction, whence we will use the notation $\mathcal{A} \vee \mathcal{B} := \Delta_{\vee}^{X}(\mathcal{A}, \mathcal{B})$ and $\mathcal{A} \wedge \mathcal{B} := \Delta_{\wedge}^{X}(\mathcal{A}, \mathcal{B})$.

Families arising from logic

Classical logic:

$$\|\phi\|^M = \{(a_0, \ldots, a_{m-1}) \in M^m \mid M \models \phi(a_0, \ldots, a_{m-1})\}.$$

For every formula ϕ, with free variables in $\vec{x} = (x_0, \ldots, x_{m-1})$, of a logic based on team semantics (i.e. for which $M \models_T \phi$ is defined for teams, sets of assignments, $T \subseteq M^k$) we have the set of teams

$$\|\phi\|^{M,\vec{x}} = \{T \subseteq M^m \mid M \models_T \phi\}.$$

The atomic level

Suppose T is a team i.e. a set of assignments s in a model M for the relevant variables.

- **Dependence atom**: $M \models_T {=}(\vec{x}, y)$ if and only if $s(\vec{x}) = s'(\vec{x})$ implies $s(y) = s'(y)$ for all $s, s' \in T$.
- We allow $\text{len}(\vec{x}) = 0$ and call ${=}(y)$ the **constancy atom**. More generally, $M \models_T {=}(\vec{y})$ if and only if $s(\vec{y}) = s'(\vec{y})$ for all $s, s' \in T$.
- **Exclusion atom**: $M \models_T \vec{x} \mid \vec{y}$ if and only if for every $s, s' \in T$ we have $s(\vec{x}) \neq s'(\vec{y})$.

- **Inclusion atom**: $M \models_T \vec{x} \subseteq \vec{y}$ if and only if for every $s \in T$ there is $s' \in T$ such that $s(\vec{x}) = s'(\vec{y})$.
- **Anonymity atom**: $M \models_T \vec{x} \Upsilon y$ if and only if for every $s \in T$ there is $s' \in T$ such that $s(\vec{x}) = s'(\vec{x})$ and $s(y) \neq s'(y)$.
- **Independence atom**: $M \models_T \vec{x} \perp_{\vec{z}} \vec{y}$ if and only if for every $s, s' \in T$ such that $s(\vec{z}) = s'(\vec{z})$ there is $s'' \in T$ such that $s''(\vec{z}) = s(\vec{z})$, $s''(\vec{x}) = s(\vec{x})$ and $s''(\vec{y}) = s'(\vec{y})$. The atom $\vec{x} \perp \vec{y}$, corresponding to the case $\vec{z}$ is empty, is called the *pure* independence atom, while $\vec{x} \perp_{\vec{z}} \vec{y}$ is otherwise called the *conditional* independence atom.

- If ϕ is a dependence atom or an exclusion atom, then $\|\phi\|^{M,\vec{x}}$ is downward closed but not necessarily closed under unions or dominated.
- If ϕ is an inclusion atom or an anonymity atom, then $\|\phi\|^{M,\vec{x}}$ is closed under unions and dominated by $M^{\mathrm{len}(\vec{x})}$ but not necessarily downward closed.

We recall the inductive definition of $M \models_T \phi$ for composite ϕ from [Vää07].

- If $a \in M$, then $s(a/x)$ is the unique assignment s' such that $s'(x) = a$ and $s'(y) = s(y)$ for variables y in the domain of s other than x.
- If $F : T \to \mathcal{P}(M) \setminus \{\emptyset\}$, then

$$T[F/x] = \{s(a/x) \mid s \in T, a \in F(s)\}$$

$$T[M/x] = \{s(a/x) \mid a \in M, s \in T\}.$$

Logical operations

Definition

(a) $M \models_T \phi$, where ϕ is (first order) atomic or negated atomic if and only if every assignment s in T satisfies ϕ.

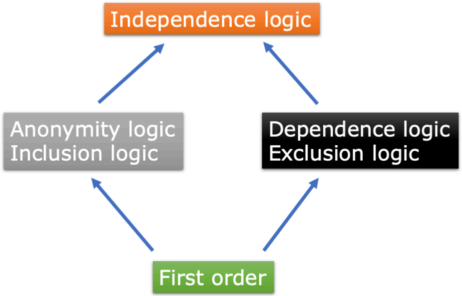
(b) $M \models_T \phi \wedge \psi$ if and only if $M \models_T \phi$ and $M \models_T \psi$.

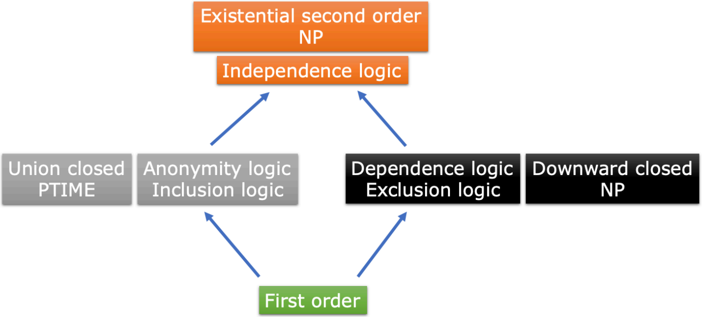
(c) $M \models_T \phi \vee \psi$ if and only if $T = U \cup V$ such that $M \models_U \phi$ and $M \models_V \psi$. (Tensor disjunction)

(d) $M \models_T \exists x \phi$ if and only if there is $F : T \to \mathcal{P}(M) \setminus \{\emptyset\}$ such that $M \models_{T[F/x]} \phi$.

(e) $M \models_T \forall x \phi$ if and only if $M \models_{T[M/x]} \phi$.

New atom	New logic $(\vee, \wedge, \forall, \exists)$	
$=\!(x, y)$ $x \mid y$	Dependence logic $=$ Exclusion logic	$\downarrow$-closed NP
$x \Upsilon y$ $x \subseteq y$	Anonymity logic $=$ Inclusion logic	P on o. f.
$x \perp y$ $x \perp_z y$	Independence logic$=$ Cond. indep. logic	NP


Independence logic
Anonymity logic
Inclusion logic
Dependence logic
Exclusion logic
First order


Existential second order
NP
Independence logic
Union closed
PTIME
Anonymity logic
Inclusion logic
Dependence logic
Exclusion logic
Downward closed
NP
First order

For every (classical) first order formula ϕ we have

$$\|\phi\|^{M,\vec{x}} = [\emptyset, T_\phi] = \mathcal{P}(T_\phi),$$

where $T_\phi = (\|\phi\|^M =)\{\vec{a} \in M^m \mid M \models \phi(\vec{a})\}$. Thus for first order ϕ the family $\|\phi\|^{M,\vec{x}}$ is dominated (by T_ϕ), downward closed, and convex.

Operators at work

$$\begin{aligned}
\|\phi \wedge \psi\|^{M,\vec{x}} &= \|\phi\|^{M,\vec{x}} \cap \|\psi\|^{M,\vec{x}} \\
\|\phi \vee \psi\|^{M,\vec{x}} &= \|\phi\|^{M,\vec{x}} \vee \|\psi\|^{M,\vec{x}} \\
\|\exists x_i \phi\|^{M,\vec{x}^-} &= \Delta^{M^m}_{\exists i}(\|\phi\|^{M,\vec{x}}) \\
\|\forall x_i \phi\|^{M,\vec{x}^-} &= \Delta^{M^m}_{\forall i}(\|\phi\|^{M,\vec{x}}),
\end{aligned}$$

where $\vec{x}^-$ is the tuple obtained from $\vec{x}$ by deleting the component x_i.

Towards combinatorics of the atoms

For non-empty finite sets X and Y, here is a list of families that we consider:

$$\begin{aligned}
\mathcal{F} &= \{f \subseteq X \times Y \mid f \text{ is a mapping } \},\\
\mathcal{X} &= \{R \subseteq X \times X \mid \mathrm{dom}(R) \cap \mathrm{rg}(R) = \emptyset\}\\
\mathcal{I}_{\subseteq} &= \{R \subseteq X \times X \mid \mathrm{dom}(R) \subseteq \mathrm{rg}(R)\},\\
\mathcal{Y} &= \{R \subseteq X \times Y \mid R \text{ is anonymous}\},\\
\mathcal{I}_{\perp} &= \{A \times B \mid A \subseteq X,\ B \subseteq Y\},
\end{aligned}$$

where we call a relation $R \subseteq X \times Y$ *anonymous* if for all $x \in \mathrm{dom}(R)$ there exist distinct $y, y' \in Y$ with $(x, y), (x, y') \in R$.

Dimension computations

Theorem

Let X and Y be finite sets with $\ell = |X| \geq 2$ and $n = |Y| \geq 2$. Then:

$$
\begin{aligned}
\mathrm{D}(\mathcal{F}) &= n^{\ell} \\
\mathrm{D}(\mathcal{X}) &= 2^{\ell} - 2 \\
\mathrm{D}(\mathcal{I}_{\subseteq}) &= 2^{\ell} - \ell \\
\mathrm{D}(\mathcal{Y}) &= 2^{\ell} \\
\mathrm{D}(\mathcal{I}_{\perp}) &= (2^{\ell} - \ell - 1)(2^{n} - n - 1) + \ell + n
\end{aligned}
$$

Accordingly...

$x = y$	1	
$=(\vec{y})$	n^m	$\text{len}(\vec{y}) = m$
$\vec{x} \subseteq \vec{y}$	$2^{n^m} - n^m$	$\text{len}(\vec{x}) = \text{len}(\vec{y}) = m$
$\vec{x} \mid \vec{y}$	$2^{n^m} - 2$	$\text{len}(\vec{x}) = \text{len}(\vec{y}) = m$
$\vec{x} \Upsilon y$	2^{n^m}	$\text{len}(\vec{x}) = m$
$\vec{x} \perp \vec{y}$	$\approx 2^{n^m + n^k}$	$\text{len}(\vec{x}) = m, \text{len}(\vec{y}) = k$
$=(\vec{x}, y)$	n^{n^m}	$\text{len}(\vec{x}) = m$
$\vec{x} \perp_{\vec{u}} \vec{y}$	$\approx [2^{n^m + n^k}, 2^{n^{m+s} + n^{k+s}}]$	$\text{len}(\vec{x}) = m, \text{len}(\vec{y}) = k, \text{len}(\vec{u}) = s$

Table: Dimensions of atoms.

Definition

A set $\mathbb{O}$ of mappings $f\colon \mathbb{N} \to \mathbb{N}$ is a *growth class* if the following conditions hold for all $f, g\colon \mathbb{N} \to \mathbb{N}$:

(a) If $g \in \mathbb{O}$ and $f \leq g$, then $f \in \mathbb{O}$.

(b) If $f, g \in \mathbb{O}$, then $f + g \in \mathbb{O}$ and $fg \in \mathbb{O}$.

- We are interested in the following particular classes: For $k \in \mathbb{N}$, the class $\mathbb{E}_k$ consist all $f : \mathbb{N} \to \mathbb{N}$ such that there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ of degree k and with coefficients in $\mathbb{N}$ such that $f(n) \leq 2^{p(n)}$.
- $\mathbb{F}_k$ is the class of functions $f : \mathbb{N} \to \mathbb{N}$ such that there exists a polynomial $p : \mathbb{N} \to \mathbb{N}$ of degree k and with coefficients in $\mathbb{N}$ such that $f(n) \leq n^{p(n)}$.

Note that $\mathbb{E}_0$ is the class of bounded functions and $\mathbb{F}_0$ the class of functions of polynomial growth. The following is immediate:

Theorem

Each $\mathbb{E}_k$ and $\mathbb{F}_k$ (for $k \in \mathbb{N}$) is a growth class. Furthermore, we have that

$$\mathbb{E}_0 \subsetneq \mathbb{F}_0 \subsetneq \mathbb{E}_1 \subsetneq \mathbb{F}_1 \subsetneq \cdots \subsetneq \mathbb{E}_k \subsetneq \mathbb{F}_k.$$

Definition

To each formula ϕ with free variables in $\vec{x}$ allowing a team-semantical interpretation we relate the following *dimension function* $\mathrm{Dim}_{\phi,\vec{x}} \colon \mathbb{N} \to \mathrm{Card}$:

$$\mathrm{Dim}_{\phi,\vec{x}}(n) = \sup \left\{ \mathrm{D}(\|\phi\|^{M,\vec{x}}) \mid M \text{ is a model}, |M| = n \right\}.$$

1. $\mathrm{Dim}_{\phi,\vec{x}}(n) = 1$, hence $\mathrm{Dim}_{\phi,\vec{x}}$ is in $\mathbb{E}_0$, for every first order ϕ.
2. $\mathrm{Dim}_{=(\vec{x},y),\vec{x}y}(n) = n^{n^k}$, hence $\mathrm{Dim}_{=(\vec{x},y),\vec{x}y}$ is in $\mathbb{F}_k$, where $\mathrm{len}(\vec{x}) = k$.
3. $\mathrm{Dim}_{\vec{x}|\vec{y},\vec{x}\vec{y}}(n) = 2^{n^k} - 2$, hence $\mathrm{Dim}_{\vec{x}|\vec{y},\vec{x}\vec{y}}$ is in $\mathbb{E}_k$, where $\mathrm{len}(\vec{x}) = \mathrm{len}(\vec{y}) = k$.
4. $\mathrm{Dim}_{\vec{x}\subseteq\vec{y},\vec{x}\vec{y}}(n) = 2^{n^k} - n^k$, hence $\mathrm{Dim}_{\vec{x}\subseteq\vec{y},\vec{x}\vec{y}}$ is in $\mathbb{E}_k$, where $\mathrm{len}(\vec{x}) = \mathrm{len}(\vec{y}) = k$.
5. $\mathrm{Dim}_{\vec{x}\Upsilon y,\vec{x}y}(n) = 2^{n^k}$, hence $\mathrm{Dim}_{\vec{x}\Upsilon y,\vec{x}y} \in \mathbb{E}_k$, where $\mathrm{len}(\vec{x}) = k$.
6. $\mathrm{Dim}_{\vec{x}\perp_{\vec{z}}\vec{y},\vec{x}\vec{z}\vec{y}}(n) \in [r, r^{n^s}]$, where $r = (2^{n^m} - n^m - 1)(2^{n^k} - n^k - 1) + n^m + n^k$, hence $\mathrm{Dim}_{\vec{x}\perp_{\vec{z}}\vec{y},\vec{x}\vec{z}\vec{y}}$ is in $\mathbb{E}_{m+k+s}$, where $\mathrm{len}(\vec{x}) = k$, $\mathrm{len}(\vec{y}) = m$, and $\mathrm{len}(\vec{z}) = s$.

family	X	Y	Z	formula ϕ	Dim_α
$\mathcal{F}$	M^k	M		$=(\vec{x}, t)$	$\mathbb{F}_k$
$\mathcal{X}$	M^k	M^k		$\vec{x} \mid \vec{y}$	$\mathbb{E}_k$
$\mathcal{I}_\subseteq$	M^k	M^k		$\vec{x} \subseteq \vec{y}$	$\mathbb{E}_k$
$\mathcal{Y}$	M^k	M^l		$\vec{x} \Upsilon y$	$\mathbb{E}_k$
$\mathcal{I}_\perp$	M^k	M^l		$\vec{x} \perp \vec{z}$	$\mathbb{E}_{k+l}$
$\mathcal{I}_{\perp.}$	M^k	M^l	M^s	$\vec{x} \perp_{\vec{z}} \vec{y}$	$\mathbb{E}_{m+k+s}$

Dimension under various operators

Let X and Y be nonempty base sets, and let $\mathcal{R} \subseteq \mathcal{P}(Y) \times \mathcal{P}(X)^n$ be an $(n+1)$-ary relation. Then we define a operator $\Delta_{\mathcal{R}} \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ by the condition

$$B \in \Delta_{\mathcal{R}}(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) \iff \\ \exists A_0 \in \mathcal{A}_0 \ldots \exists A_{n-1} \in \mathcal{A}_{n-1} : \ (B, A_0, \ldots, A_{n-1}) \in \mathcal{R}.$$

Definition ([Lüc20])

Let X and Y be nonempty sets. A function $\Delta \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ is a Kripke-operator, if there is a relation $\mathcal{R} \subseteq \mathcal{P}(Y) \times \mathcal{P}(X)^n$ such that $\Delta = \Delta_{\mathcal{R}}$.

- Intersection of families is a Kripke-operator: If $\mathcal{A}, \mathcal{B} \subseteq \mathcal{P}(X)$ and $C \in \mathcal{P}(X)$, then $C \in \mathcal{A} \cap \mathcal{B}$ if and only if there exist $A \in \mathcal{A}$ and $B \in \mathcal{B}$ such that $(C, A, B) \in \mathcal{R}_\cap$, where $\mathcal{R}_\cap$ is the relation $\{(D, D, D) \mid D \in \mathcal{P}(X)\}$.
- Union of families on X is *not* a Kripke-operator.
- Complementation Δ_c^X is not a Kripke-operator

- Tensor disjunction and negation on X are Kripke-operators: clearly $\mathcal{A} \vee \mathcal{B} = \Delta_{\mathcal{R}_\vee}(\mathcal{A}, \mathcal{B})$ and $\Delta_\neg^X(\mathcal{A}) = \Delta_{\mathcal{R}_\neg}(\mathcal{A})$ where $\mathcal{R}_\vee = \{(A \cup B, A, B) \mid A, B \in \mathcal{P}(X)\}$ and $\mathcal{R}_\neg = \{(X \setminus A, A) \mid A \in \mathcal{P}(X)\}$.
- Projections and inverse projections are Kripke-operators. Indeed, if $f : X \to Y$ is a surjection, then clearly $\Delta_f = \Delta_{\mathcal{R}_f}$, where $\mathcal{R}_f = \{(f[A], A) \mid A \in \mathcal{P}(X)\}$. Similarly, $\Delta_{f^{-1}} = \Delta_{\mathcal{R}_{f^{-1}}}$, where $\mathcal{R}_{f^{-1}} = \{(A, f[A]) \mid A \in \mathcal{P}(X)\}$.
- The existential quantification operators $\Delta_{\exists i}^{M^m}$ and the universal quantification operators $\Delta_{\forall i}^{M^m}$ are Kripke-operators.

Definition

Let $\Delta\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ be an operator. We say that Δ *weakly preserves dominated convexity* if $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$ is dominated and convex or $\Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1}) = \emptyset$ whenever $\mathcal{A}_i$ is dominated and convex for each $i < n$.

Theorem

Let $\Delta_{\mathcal{R}}\colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ be a Kripke-operator, and let $\mathcal{A} = \Delta(\mathcal{A}_0, \ldots, \mathcal{A}_{n-1})$. If Δ weakly preserves dominated convexity then $\mathrm{D}(\mathcal{A}) \leq \mathrm{D}(\mathcal{A}_0) \cdot \ldots \cdot \mathrm{D}(\mathcal{A}_{n-1})$.

Below we will use the notation

$$\mathcal{R}[A] := \{(A_0, \ldots, A_{n-1}) \mid (A, A_0, \ldots, A_{n-1}) \in \mathcal{R}\}.$$

Definition ([Lüc20])

A Kripke-operator $\Delta_{\mathcal{R}} \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ is *local* if, for any $A \in \mathcal{P}(Y)$, $\mathcal{R}[A]$ is determined by the relations $\mathcal{R}[\{a\}]$, $a \in A$, as follows:

$(A_0, \ldots, A_{n-1}) \in \mathcal{R}[A] \iff$ *for each* $a \in A$ *there is* $(A_0^a, \ldots, A_{n-1}^a) \in \mathcal{R}[\{a\}]$ *such that* $A_i = \bigcup_{a \in A} A_i^a$ *for* $i < n$.

Theorem

If $\Delta_{\mathcal{R}} \colon \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ *is a local Kripke-operator for finite* X *and* Y*, then it weakly preserves dominated convexity.*

Definition

A Kripke-operator $\Delta_{\mathcal{R}} : \mathcal{P}(\mathcal{P}(X))^n \to \mathcal{P}(\mathcal{P}(Y))$ is *separating* if $A_i \cap B_i = \emptyset$ for all $i < n$ whenever $(A_0, \ldots, A_{n-1}) \in \mathcal{R}[\{a\}]$, $(B_0, \ldots, B_{n-1}) \in \mathcal{R}[\{b\}]$ and $a \neq b$.

Theorem

The operators $\Delta_{\cap}^{M^m}$, $\Delta_{\vee}^{M^m}$ and $\Delta_{\mathcal{K},\vec{\ell}}^{M^m}$ are local and separating.

Hence they preserve dimension!

Corollary

Let $\mathbb{O}$ be a growth class. Furthermore, let $\phi = \phi(\vec{x})$ and $\psi = \psi(\vec{x})$ be formulas of some logic $\mathcal{L}$ with team semantics.

(a) *If* $\mathrm{Dim}_{\phi,\vec{x}}, \mathrm{Dim}_{\psi,\vec{x}} \in \mathbb{O}$, *then* $\mathrm{Dim}_{\phi\wedge\psi,\vec{x}} \in \mathbb{O}$.

(b) *If* $\mathrm{Dim}_{\phi,\vec{x}}, \mathrm{Dim}_{\psi,\vec{x}} \in \mathbb{O}$, *then* $\mathrm{Dim}_{\phi\vee\psi,\vec{x}} \in \mathbb{O}$.

(c) *If* $\mathrm{Dim}_{\phi,\vec{x}} \in \mathbb{O}$, *then* $\mathrm{Dim}_{\exists x_i\phi,\vec{x}^-} \in \mathbb{O}$ *and* $\mathrm{Dim}_{\forall x_i\phi,\vec{x}^-} \in \mathbb{O}$, *where $\vec{x}^-$ is $\vec{x}$ without the component x_i.*

(d) *If $Q_{\mathcal{K}}$ is a Lindström quantifier, $\vec{x} = \vec{z} \otimes_{\vec{\ell}} \vec{y}$ and* $\mathrm{Dim}_{\phi,\vec{x}} \in \mathbb{O}$, *then* $\mathrm{Dim}_{Q_{\mathcal{K}}\vec{y}\,\phi,\vec{z}} \in \mathbb{O}$.

Definition

The logic $\mathbb{LE}_k$ is the closure of literals and all atoms whose dimension function is in the growth class $\mathbb{E}_k$ under the connectives $\wedge$, $\vee$ and any Lindström quantifiers. Similarly $\mathbb{LF}_k$ for $\mathbb{F}_k$.

Lemma

(a) $\mathbb{LE}_k \subseteq \mathbb{LF}_k \subseteq \mathbb{LE}_{k+1} \subseteq \mathbb{LF}_{k+1}$.

The arity-concept

Definition

- The atom $=(\vec{x}, y)$ is k-ary, if $\text{len}(\vec{x}) = k$,
- The atoms $\vec{x} \mid \vec{y}$ and $\vec{x} \Upsilon y$ are k-ary if $\text{len}(\vec{x})(= \text{len}(\vec{y})) = k$,
- The atom $\vec{t}_2 \perp_{\vec{t}_1} \vec{t}_3$ is $m + \max(k, l)$-ary, or alternatively (k, l, m)-ary, if $\text{len}(\vec{t}_1) = m, \text{len}(\vec{t}_2) = k,$ and $\text{len}(\vec{t}_3) = l$.
- The atom $\vec{t}_2 \perp \vec{t}_3$ is $\max(k, l)$-ary, or alternatively (k, l)-ary, if $\text{len}(\vec{t}_2) = k,$ and $\text{len}(\vec{t}_3) = l$.

Theorem

1. *k-ary inclusion, anonymity, exclusion and independence logics are all included in $\mathbb{LE}_k$.*
2. *The k-ary dependence logic is included in $\mathbb{LF}_k$.*
3. *The (k, l, m)-ary independence logic is included in $\mathbb{LF}_{\max(k,l)+m}$.*

Theorem

(a) *The dimension of every formula in $\mathbb{LE}_k$ is in the growth class $\mathbb{E}_k$.*

(b) *The dimension of every formula in $\mathbb{LF}_k$ is in the growth class $\mathbb{F}_k$.*

Theorem

1. *The $k+1$-ary inclusion, anonymity, exclusion and independence atoms are not definable in $\mathbb{LE}_k$.*
2. *The $k+1$-ary dependence atom is not definable in $\mathbb{LF}_k$.*
3. *The (k,l,m)-ary independence atom is not definable in $\mathbb{LF}_i$ if $i < \max(k,l)+m$.*

For comparison ([Gal12]):

(a) The k-ary dependence atom is definable from the $k+1$-ary exclusion atom and also in terms of the $k+1$-ary pure independence atom, and in the other direction, the k-ary exclusion atom is definable from the k-ary dependence atom.

(b) The k-ary exclusion atom can be defined in terms of the k-ary inclusion and the k-ary pure independence atoms.

(c) The k-ary inclusion atom can be defined from the $(k,2)$-ary pure independence atom.

(d) The k-ary anonymity atom is definable in terms of the $k+1$-ary inclusion atom.

(e) The (k,l,m)-ary independence atom is definable in terms of the $k+l+m$-ary dependence atom, $k+l$-ary, $k+m$-ary exclusion atoms, and the $k+l+m$-ary inclusion atom.

(f) The (k,l,m)-ary independence atom is definable in terms of the pure $(k+m, l+m)$-ary independence atom (Wilke).

Corollary (Hierarchy Theorem)

Dependence logic, exclusion logic, inclusion logic, anonymity logic and pure independence logic each has a proper definability hierarchy for formulas based on the arity of the non-first order atoms.

- The k-ary dependence atom is not definable in the extension of first order logic by $< k$-ary dependence (or any other $< k$-ary) atoms, $\leq k$-ary independence, exclusion, inclusion, anonymity, constancy atoms, and any Lindström quantifiers.
- The k-ary exclusion atom is not definable in the extension of first order logic by $< k$-ary exclusion, inclusion, anonymity, dependence, independence, constancy (or any other $< k$-ary) atoms, and any Lindström quantifiers.
- The k-ary inclusion atom is not definable in the extension of first order logic by $< k$-ary inclusion, exclusion, anonymity, dependence, or constancy (or any other $< k$-ary) atoms, and any Lindström quantifiers.
- The k-ary anonymity atom is not definable in the extension of first order logic by $< k$-ary inclusion, anonymity, exclusion, dependence, constancy (or any other $< k$-ary) atoms, and any Lindström quantifiers.
- The k-ary independence atom (whether pure or not) is not definable in the extension of first order logic by $< k$-ary independence, inclusion, anonymity, exclusion, dependence, constancy (or any other $< k$-ary) atoms, and any Lindström quantifiers.

Many open problems:

1. Is the k-ary dependence atom definable in the extension of first order logic by k-ary independence, exclusion, inclusion, anonymity, constancy atoms, and some Lindström quantifiers?
2. Is the k-ary anonymity atom definable in terms of the k-ary inclusion atom?
3. Is the (k, l, m)-ary independence atom definable in terms of the $\max(k, l) + m$-ary dependence atom, $\max(k, l) + m$-ary, $\max(k, l) + m$-ary exclusion atoms, and the $\max(k, l) + m$-ary inclusion atom?

- Earlier hierarchy results have been for sentences.
- In [DK12] it is shown that k-ary dependence atom is weaker than $k+1$-ary dependence atom for sentences in vocabulary having arity $k+1$.
- In [Han18] it is shown (using similar results of Grohe on transitive closure and fixpoint operator) that inclusion logic with $k-1$-ary inclusion atoms is strictly weaker than inclusion logic with k-ary inclusion atoms for sentences when $k \geq 2$.
- In [GHK13] it is shown that independence logic with k-ary independence atoms is strictly weaker than independence logic with $k+1$-ary independence atoms on the level of sentences.
- See also [Rön16] for similar hierarchy results.

Other logical operations

The atoms and logical operations $\wedge$, $\vee$, $\forall$, and $\exists$ are by no means the only ones that can be or have been considered.

Definition (Intuitionistic implication)

The intuitionistic implication $\phi \to \psi$ is defined by $M \models_T \phi \to \psi$ if and only if every $Y \subseteq T$ that satisfies in M the formula ϕ satisfies also the formula ψ.

As the following lemma demonstrates, the dependence atom can be defined in terms of the constancy atoms and the intuitionistic implication:

Lemma ([AV09])

$\models =\!(x_1, \ldots, x_n, y) \equiv (=\!(x_1) \wedge \ldots \wedge =\!(x_n)) \ \rightarrow =\!(y)$

This gives an example where the use of $\phi \rightarrow \psi$ leads to something we know is exponential. It shows that we cannot hope to prove that the dimensions of $\phi \rightarrow \psi$ is in general better than exponential in the dimensions of ϕ and ψ.

We can add intuitionistic implication to F_0, because it does not increase dimension, when the latter is 1.

Definition (Intuitionistic disjunction)

$M \models_T \phi \veebar \psi$ if and only if $M \models_T \phi$ or $M \models_T \psi$.

Note:

$$\|\phi \veebar \psi\|^{M,\vec{x}} = \|\phi\|^{M,\vec{x}} \cup \|\psi\|^{M,\vec{x}}.$$

Intuitionistic disjunction can be defined in terms of constancy atoms:

$$\models \phi \veebar \psi \iff \exists x \exists y (=\!(x) \wedge =\!(y) \wedge ((x = y \wedge \phi) \vee (\neg x = y \wedge \psi))).$$

But since it increases dimension additively, it cannot be defined in first order logic alone. In fact, the formula $x = y \veebar \neg x = y$ has dimension 2.

Definition

- If $a \in M$, let F_a be the constant function $F_a(s) = a$ for all $s \in T$.
- The $\exists^1$-quantifier is defined as follows: $M \models_T \exists^1 x\phi$ if for some $a \in M$ we have $M \models_{T[F_a/x]} \phi$.
- The $\forall^1$-quantifier is defined as follows: $M \models_T \forall^1 x\phi$ if for all $a \in M$ we have $M \models_{T[F_a/x]} \phi$.
- The *public announcement*-quantifier ([Gal12]) $\delta^1 x$ is defined as follows: $M \models_T \delta^1 x\phi$ if for all $a \in M$ we have $M \models_{T_a} \phi$, where $T_a = \{s \in T : s(x) = a\}$.

Lemma ([Gal12])

(a) $\models \forall^1 x\phi(x) \iff \forall x(=(x) \to \phi(x))$

(b) $\models \delta^1 x\phi(x) \iff \forall^1 y(x \neq y \vee \phi(x))$

(c) $\models \forall^1 x\phi(x) \iff \forall x\delta^1 x\phi(x)$

(d) $\models =(x_1, \ldots, x_n, y) \iff \delta^1 x_1 \ldots \delta^1 x_n =(y)$

- This shows that we cannot hope to prove that they are in general better than exponential.
- This also shows that these operators do not arise from a Lindström quantifier.
- Note that by iterating $\forall^1 x$ or $\delta^1 x$ we can defined dependence atoms of arbitrary arity.
- This shows that $\forall^1 x$ and $\delta^1 x$ increase dimension more than any k-ary atom for a fixed k.

Lemma

- $\models \exists^1 x\phi \iff \exists x(=(x) \wedge \phi)$.
- $\models =(x) \iff \exists^1 y(x = y)$.
- $\exists^1$ *increases dimension at most linearly.*
- $\exists^1$ *does indeed increase dimension, as the dimension of* $x = y$ *is 1 and the dimension of* $=(x)$ *is* n.

Definition ([Gal12])

A generalized quantifier (which need not be a Lindström quantifier) Q of a logic L_1 is said to be *uniformly definable* in another logic L_2 if the logic L_2 has a sentence $\Phi(P)$, P unary, with only positive occurrences of P, such that for all formulas $\phi(x, y)$ of the logic L_1 we have

$$\models Qx\phi(x, y) \iff \Phi(\phi(z, y)/P(z)).$$

Similarly, if there are several formulas, as in $Qxy\phi(x, z)\psi(y, z)$.

In first order logic definability is always uniform.

Example

The quantifier $\exists^1$ is uniformly definable in dependence logic:

$$\models \exists^1 x\phi(x,y) \iff \exists x(=(x) \wedge \phi(x,y))$$

.

The intuitionistic disjunction is uniformly definable in dependence logic:

$$\models \phi \veebar \psi \iff \exists x \exists y(=(x) \wedge =(y) \wedge ((x=y \wedge \phi) \vee (\neg x = y \wedge \psi))).$$

Lemma

Suppose $\models Qx\phi(x,y) \iff \Phi(\phi(z,y)/P(z))$ *where* $\Phi(P)$ *is a sentence in dependence logic. Then*

$$\mathrm{Dim}_{Qx\phi(x,y),xy}(n) \leq (n^{n^m} \cdot \mathrm{Dim}_{\phi(x,y)}(n))^k,$$

where k *is the length of* $\Phi(P)$ *and* m *is the maximum of the lengths of* $\vec{x}$ *such that* $=\!(\vec{x}, y)$ *for some* y *occurs in* $\Phi(P)$.

Corollary ([Gal12])

The quantifier $\forall^1$ *is not uniformly definable in dependence logic.*

Summary

- With our dimension concept one can prove hierarchy results for formulas, not just sentences.
- Dimension reveals subtle qualitative differences between logical operations (cf. $\forall^1, \rightarrow, \underline{\vee}$).
- Our method is very general, applies to arbitrary families of sets in a finite domain.

Thank you!

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